

# Complex weak values in quantum measurement

Richard Jozsa

*Department of Computer Science, University of Bristol,  
Merchant Venturers Building, Bristol BS8 1UB U.K.*

## Abstract

In the weak measurement formalism of Y. Aharonov et al. the so-called weak value  $A_w$  of any observable  $A$  is generally a complex number. We derive a physical interpretation of its value in terms of the shift in the measurement pointer's mean position and mean momentum. In particular we show that the mean position shift contains a term jointly proportional to the imaginary part of the weak value and the rate at which the pointer is spreading in space as it enters the measurement interaction.

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## 1 Introduction

In quantum mechanics the essential connection between theory and experimental outcomes may be thought of as being embodied in the formula

$$\langle A \rangle = \langle \psi_i | A | \psi_i \rangle \quad (1)$$

for the measured mean value of an observable  $A$  upon (strong) measurement of a quantum system prepared in state  $|\psi_i\rangle$ . The formalism of weak measurement [2] developed by Y. Aharonov and co-workers (c.f. [1] chapters 16,17 for a review and further references therein) provides an alternative foundation for quantum measurement theory. In this formalism the above formula becomes replaced [2] by a more general expression:

$$A_w = \frac{\langle \psi_f | A | \psi_i \rangle}{\langle \psi_f | \psi_i \rangle}. \quad (2)$$

$A_w$  is called the *weak value* of observable  $A$  for a quantum system pre-selected in state  $|\psi_i\rangle$  and post-selected in state  $|\psi_f\rangle$  and it characterises the observed outcomes of weak measurements. If  $|\psi_i\rangle$  is an eigenstate of  $A$  then  $\langle A \rangle$  and  $A_w$  agree (both equalling the corresponding eigenvalue) but more generally  $A_w$  need not lie within the range of eigenvalues and may even be complex. Thus its significance is more subtle than the straightforward interpretation of  $\langle A \rangle$  as a measured mean value. In this note we establish a physical interpretation for  $A_w$  in its most general context.

The formalism of weak measurement has two ingredients that differ from the usual approach that leads to eq. (1): firstly in addition to preparation of quantum systems in a given initial state we also impose *post-selection* into a given final state; secondly we consider a scenario in which the measurement interaction is suitably weak so that after measurement the system state is left largely intact. As a framework for our main results we begin by

briefly reviewing the weak measurement formalism and the origin of the expression  $A_w$  in eq. (2).

Consider a quantum system prepared in state  $|\psi_i\rangle$  upon which we wish to measure  $A$ . For measurement process we use the standard von Neumann paradigm [3] introducing a pointer in initial state  $|\phi\rangle$  with wavefunction  $\phi(q)$ , and interaction hamiltonian

$$H_{\text{int}} = g(t)Ap \quad g(t) = g\delta(t - t_0) \quad (3)$$

where  $g$  is a coupling constant and  $p$  is the pointer momentum conjugate to the position co-ordinate  $q$ . Here we have taken the interaction to be impulsive at time  $t = t_0$  (and the expression  $Ap$  is shorthand for  $(A \otimes I)(I \otimes p)$ , where the first and second slots refer to the system and pointer respectively.)

After interaction the system and pointer are in joint state  $e^{-igAp} |\psi_i\rangle |\phi\rangle$  and we post-select the system on state  $|\psi_f\rangle$  resulting in the (sub-normalised) pointer state

$$|\alpha\rangle = \langle\psi_f| e^{-igAp} |\psi_i\rangle |\phi\rangle. \quad (4)$$

(Here and hereafter we adopt units making  $\hbar = 1$ ). In practice the post-selection is achieved by running the process many times with initial state  $|\psi_i\rangle$  and after all tasks are completed we perform a further final measurement of the projector  $\Pi_f$  onto  $|\psi_f\rangle$  in each run. Then for statistical analysis of measurement outcomes or any other considerations, we retain only those runs for which  $\Pi_f$  yielded 1. (The sub-normalisation of  $|\alpha\rangle$  reflects the probability of success in this  $\Pi_f$  measurement). It is a remarkable fact that quantum theory allows both pre- and post-selection of systems whereas classical physics allows imposition of only either initial or final boundary conditions, but not both (cf [1] §16.3).

It is a standard tenet of quantum theory that measurement irrevocably disturbs a quantum system. The measurement interaction eq. (3) is said to be strong if the translated wavefunctions  $\phi(q - ga_i)$  for eigenvalues  $a_i$  of  $A$ , correspond to states that have negligible overlap. In that case after the measurement interaction the pointer position will be observed, on average, to have shifted by  $g\langle A \rangle$ . In contrast to this standard scenario, the second basic ingredient in the formalism of weak measurement is the requirement that the measurement interaction eq. (3) be suitably weak so that we may obtain information about  $A$  while the system state is left largely intact. To restrict the strength of interaction we consider the limit of small  $g$ , retaining only terms to first order in  $g$ . Alternatively weakness may be imposed by requiring  $p$  to remain small which, by the  $\Delta p \Delta q$  uncertainty relation, corresponds to a limit of increasingly broad initial wavefunctions of the pointer in the  $q$  representation. In the following we will work only with the limit of small  $g$ . In both cases the translates  $\phi(q - ga_i)$  will retain a large overlap (of size  $1 - O(g)$  or  $1 - O(p)$ ). Expanding eq. (4) to terms of  $O(g)$  yields:

$$\begin{aligned} |\alpha\rangle &\approx \langle\psi_f| I - igAp |\psi_i\rangle |\phi\rangle \\ &= \langle\psi_f|\psi_i\rangle (I - igA_w p) |\phi\rangle \end{aligned} \quad (5)$$

$$\approx \langle\psi_f|\psi_i\rangle e^{-igA_w p} |\phi\rangle. \quad (6)$$

Thus it is clear that all subsequent measurement properties of the pointer depend on the ingredients  $A$ ,  $|\psi_i\rangle$  and  $|\psi_f\rangle$  only through the single c-number  $A_w$ .

From eq. (2) we see that  $A_w$  can generally be a complex number and its effect on mean values of pointer variables, such as the mean position and mean momentum, is not immediately clear from eqs. (5,6). Mathematically eq. (6) simply represents a translation  $\phi(q - gA_w)$  of the wavefunction by  $gA_w$ . However as the latter is generally complex and we use the resulting translated function only along the real  $q$  axis, its quantum mean properties are now not simply characterisable in terms of translates of those of  $\phi(q)$ . In the literature only some special restricted cases have been considered. Introduce the initial and final pointer means:

$$\langle q \rangle_i = \langle \phi | q | \phi \rangle \quad \langle q \rangle_f = \frac{\langle \alpha | q | \alpha \rangle}{\langle \alpha | \alpha \rangle} \quad (7)$$

and similarly the momentum means  $\langle p \rangle_i$  and  $\langle p \rangle_f$  with  $p$  replacing  $q$  in the above. Also introduce the variances of position and of momentum in the initial pointer state:

$$Var_q = \langle \phi | q^2 | \phi \rangle - \langle \phi | q | \phi \rangle^2 \quad Var_p = \langle \phi | p^2 | \phi \rangle - \langle \phi | p | \phi \rangle^2. \quad (8)$$

Then the following cases have been noted [1, 4]. (i) if  $A_w$  is *real* then  $\langle q \rangle_f = \langle q \rangle_i + gA_w$ ; (ii) if  $A_w$  is complex but the pointer wavefunction  $\phi(q)$  is *real-valued* then  $\langle q \rangle_f = \langle q \rangle_i + gRe(A_w)$  and  $\langle p \rangle_f = \langle p \rangle_i + 2gIm(A_w)Var_p$ ; (iii) it has also been noted ([1] p.237) that in the expression eq. (6), the imaginary part of  $A_w$  contributes a non-unitary operation which can thus be thought of as increasing or decreasing the size  $\langle \alpha | \alpha \rangle$  of the pre- and post-selected ensemble of runs.

## 2 Complex weak values

We now consider the most general case of complex  $A_w$  and complex-valued wavefunction  $\phi(q)$ . Our resulting general formulae will display a novel role for the imaginary part of  $A_w$  in the shift of pointer mean position. We will demonstrate the following.

**Theorem.** Let  $A_w = a + ib$ . Then after a weak von Neumann measurement interaction on a system with pre- and post-selected states  $|\psi_i\rangle$  and  $|\psi_f\rangle$ , the mean pointer position and momentum satisfy

$$\langle q \rangle_f = \langle q \rangle_i + ga + gb(m \frac{d}{dt} Var_q) \quad (9)$$

$$\langle p \rangle_f = \langle p \rangle_i + 2gb(Var_p) \quad (10)$$

Here  $m$  is the mass of the pointer and  $\frac{d}{dt}Var_q$  is the time derivative of its position variance as  $t \rightarrow t_0$ , the time of the impulsive measurement interaction.  $\square$

Thus in particular there is a contribution to the pointer's mean position shift that is proportional to the imaginary part of  $A_w$  and the rate at which the pointer is spreading in space as it enters the interaction.

To derive eq. (9) we begin by substituting  $p = -i\partial/\partial q$  into eq. (5). Retaining only terms to  $O(g)$  we get

$$\alpha(q)\bar{\alpha}(q) = |\langle \psi_f | \psi_i \rangle|^2 [\phi\bar{\phi} - ga(\phi'\bar{\phi} + \bar{\phi}'\phi) - igb(\phi'\bar{\phi} - \bar{\phi}'\phi)] \quad (11)$$

where  $\phi'$  denotes the space derivative and  $\bar{\phi}$  denotes the complex conjugate. The coefficient of  $ga$  is the space derivative of the probability density  $\phi\bar{\phi}$  whereas the coefficient of  $gb$  is

recognised as the spatial part of the conserved probability current for  $|\phi\rangle$ . To exploit these features we introduce

$$\phi = Re^{iS} \quad \rho = R^2. \quad (12)$$

Then

$$\alpha\bar{\alpha} = |\langle\psi_f|\psi_i\rangle|^2 [\rho - ga\rho' + gb(2\rho S')]$$

and a straightforward calculation to  $O(g)$  gives (writing  $\mu = \langle q \rangle_i$ ):

$$\langle q \rangle_f = \frac{\int \bar{\alpha} q \alpha}{\int \bar{\alpha} \alpha} = \mu - ga \int q \rho' + gb \int 2\rho S' (q - \mu) \quad (13)$$

where the integration is over all space. Integration by parts gives:

$$\langle q \rangle_f = \langle q \rangle_i + ga - gb \int (q - \mu)^2 (\rho S')'. \quad (14)$$

Now consider the Schrödinger equation of the pointer up to time  $t_0$  of interaction:

$$i\phi_t = -\frac{1}{2m}\phi'' + V(q)\phi.$$

Substituting eq. (12) and taking the imaginary part of the resulting equation gives the continuity equation for probability conservation:

$$\rho_t + \left(\rho \frac{S'}{m}\right)' = 0. \quad (15)$$

Hence  $(\rho S')' = -m\rho_t$  and eq. (14) finally gives

$$\langle q \rangle_f = \langle q \rangle_i + ga + gb(m \frac{d}{dt} Var_q) \quad (16)$$

as claimed.

We note that a wavefunction is (instantaneously) real valued (up to an overall constant phase) if and only if  $S' = 0$  and then  $dVar_q/dt$  is zero (via eq. (15) giving  $\rho_t = 0$ ) so we regain the previously quoted results (i) and (ii) for the change in  $\langle q \rangle$  in these restricted cases.

Next we present an alternative Heisenberg representation derivation of eq. (16) which generalises immediately to other pointer observables (such as  $p$ ) replacing  $q$ . Let  $M$  be any pointer observable. From eq. (5) we get to  $O(g)$ :

$$\begin{aligned} \langle M \rangle_f &= \frac{\langle \alpha | M | \alpha \rangle}{\langle \alpha | \alpha \rangle} \\ &= \frac{\langle \phi | M | \phi \rangle - igA_w \langle \phi | Mp | \phi \rangle + ig\bar{A}_w \langle \phi | pM | \phi \rangle}{\langle \phi | \phi \rangle - igA_w \langle \phi | p | \phi \rangle + ig\bar{A}_w \langle \phi | p | \phi \rangle} \\ &= \langle M \rangle_i + ig a \langle pM - Mp \rangle_i + gb (\langle pM + Mp \rangle_i - 2\langle p \rangle_i \langle M \rangle_i) \end{aligned} \quad (17)$$

(where for any observable  $N$ ,  $\langle N \rangle_i = \langle \phi | N | \phi \rangle$  is its mean value in state  $|\phi\rangle$ ).

For  $M = q$  we have the commutation relations

$$[p, q] = -i$$

and the Heisenberg equations of motion (with  $H = p^2/2m + V(q)$ ):

$$\begin{aligned} i\frac{d}{dt}\langle q \rangle &= \langle [q, H] \rangle = \frac{i\langle p \rangle}{m} \\ i\frac{d}{dt}\langle q^2 \rangle &= \langle [q^2, H] \rangle = \frac{i\langle pq+qp \rangle}{m}. \end{aligned}$$

Substitution of these into eq. (17) immediately gives eq. (16).

If instead we set  $M = p$  then  $pM - Mp$  in the coefficient of  $ga$  in eq. (17) becomes zero and the coefficient of  $gb$  becomes  $2\langle p^2 \rangle_i - 2\langle p \rangle_i = 2\text{Var}_p$  giving

$$\langle p \rangle_f = \langle p \rangle_i + 2gb\text{Var}_p$$

as claimed in the theorem. Note that the pointer observable  $p$  commutes with the measurement interaction hamiltonian  $gAp$  so this shift in  $\langle p \rangle$  is an artefact of post-selection rather than a quantum dynamical effect, in contrast to the more interesting case of the shift in  $\langle q \rangle$ .

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